Volatility-of-Volatility: A simple model-free motivation

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Abstract

Our goal is to provide a simple, intuitive and model-free motivation for the importance of volatility-of-volatility in pricing certain kinds of exotic and structured products.

KEYWORDS: volatility of volatility, exotic options, structured products.

1 Introduction

It is intuitively clear that for exotic products which are strongly dependent on the *dynamics* of the volatility surface proper modeling of the volatility-of-volatility is critical. Several authors, including Schoutens et al. (2004), Gatheral (2006) and Bergomi (2005, 2008), have shown that the same exotic product can have significantly different valuations under different stochastic volatility models.

In this short article, we want to illustrate the importance of the volatility-of-volatility without referring to any of the standard models from the literature. We compare the pricing of a couple of fundamental payoffs *with* and *without* volatility-of-volatility.

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2 A model free motivation

Let us begin by recalling the important payoff spanning formula, first observed in Breeden, Litzenberger (1978). Any twice differentiable function $H : (0, \infty) \to \mathbb{R}$ satisfies, for any $x_0 > 0$:

$$H(x) = H(x_0) + \frac{\partial H}{\partial x}(x_0) \cdot (x - x_0) + \int_0^{x_0} \frac{\partial^2 H}{\partial x^2}(K) \cdot (K - x)_+ \, dK + \int_{x_0}^{\infty} \frac{\partial^2 H}{\partial x^2}(K) \cdot (x - K)_+ \, dK \quad (1)$$

This can be generalized to less smooth payoff functions H in several ways. For example, if H is twice differentiable on $(0, \infty) \backslash \{x_0\}$, continuous at x_0 with left and right first derivatives $\frac{\partial H^-}{\partial x}(x_0)$, $\frac{\partial H^+}{\partial x}(x_0)$, the spanning formula becomes

$$\begin{aligned} H(x) &= H(x_0) - \frac{\partial H^-}{\partial x}(x_0) \cdot (x_0 - x)_+ + \frac{\partial H^+}{\partial x}(x_0) \cdot (x - x_0)_+ \\ &+ \int_0^{x_0} \frac{\partial^2 H}{\partial x^2}(K) \cdot (K - x)_+ \, dK + \int_{x_0}^{\infty} \frac{\partial^2 H}{\partial x^2}(K) \cdot (x - K)_+ \, dK \quad (2) \end{aligned}$$

More generally, the spanning formula can be extended to convex H using generalized derivatives. For our purposes, in this article, statements (1) and (2) will suffice.

In what follows, we fix two future dates $0 < T_1 < T_2$. Suppose we want to value a contract whose payoff at time T_2 is

$$\frac{1}{T_2 - T_1} \cdot \log^2\left(\frac{S_{T_2}}{S_{T_1}}\right)$$

where we have denoted by S the price of some underlying asset. We first consider the value of this contract at the *future* time T_1. From the standpoint of time T_1, this payoff can be spanned into a portfolio of vanilla options. Specifically, if we take $H(x) = \frac{1}{T_2 - T_1} \cdot \log^2\left(\frac{x}{S_{T_1}}\right)$ and use

$$\begin{aligned} \frac{\partial H}{\partial x}(x) &= \frac{2}{x \cdot (T_2 - T_1)} \log\left(\frac{x}{S_{T_1}}\right) \\ \frac{\partial^2 H}{\partial x^2}(x) &= \frac{2}{x^2 \cdot (T_2 - T_1)} \left(1 - \log\left(\frac{x}{S_{T_1}}\right)\right) \end{aligned}$$

an application of the spanning formula (1) gives

$$\begin{aligned} \frac{1}{T_2 - T_1} \log^2\left(\frac{S_{T_2}}{S_{T_1}}\right) &= \int_0^{S_{T_1}} \frac{2}{K^2 \cdot (T_2 - T_1)} \left(1 - \log\left(\frac{K}{S_{T_1}}\right)\right) \cdot (K - x)_+ \, dK \\ &+ \int_{S_{T_1}}^{\infty} \frac{2}{K^2 \cdot (T_2 - T_1)} \left(1 - \log\left(\frac{K}{S_{T_1}}\right)\right) \cdot (x - K)_+ \, dK \end{aligned}$$

Assuming European Put and Call options, of all strikes $K > 0$, are tradeable in the market, we obtain that the value of the contract at time T_1 is given by

$$
\begin{aligned}
V_{T_1}^H &= \int_0^{S_{T_1}} \frac{2}{K^2 \cdot (T_2 - T_1)} \left(1 - \log\left(\frac{K}{S_{T_1}}\right)\right) \cdot P(S_{T_1}, K, T_2 - T_1) dK \\
&+ \int_{S_{T_1}}^{\infty} \frac{2}{K^2 \cdot (T_2 - T_1)} \left(1 - \log\left(\frac{K}{S_{T_1}}\right)\right) \cdot C(S_{T_1}, K, T_2 - T_1) dK
\end{aligned}
$$

where we assume the market option prices $P(S_{T_1}, K, T_2 - T_1)$ and $C(S_{T_1}, K, T_2 - T_1)$ are such that the two integrals converge. Making the change of variable $K = S_{T_1} \cdot x$ and using the Black-Scholes pricing function we can write

$$
\begin{aligned}
P(S_{T_1}, K, T_2 - T_1) &= S_{T_1} \cdot P^{BS}(1, x; \hat{\sigma}(x), T_2 - T_1) \\
C(S_{T_1}, K, T_2 - T_1) &= S_{T_1} \cdot C^{BS}(1, x; \hat{\sigma}(x), T_2 - T_1)
\end{aligned}
$$

where we denoted by $\hat{\sigma}(x)$ the Black-Scholes implied volatility for moneyness $x = \frac{K}{S_{T_1}}$. We finally obtain the value, at time T_1, as

$$
\begin{aligned}
V_{T_1}^H &= \int_0^1 \frac{2}{x^2 \cdot (T_2 - T_1)} (1 - \log(x)) \cdot P^{BS}(1, x; \hat{\sigma}(x), T_2 - T_1)\, dx \\
&+ \int_1^{\infty} \frac{2}{x^2 \cdot (T_2 - T_1)} (1 - \log(x)) \cdot C^{BS}(1, x; \hat{\sigma}(x), T_2 - T_1)\, dx \quad (3)
\end{aligned}
$$

Note that, for our contract, its future value at time T_1 depends *only* on the volatility-by-moneyness curve (i.e. the smile) $\hat{\sigma}(x)$ (of maturity $\Delta T = T_2 - T_1$) that will prevail in the market at time T_1. Of course, at present, we do not know what ΔT-smile will prevail in the market at time T_1. Therefore, the valuation of this product will depend entirely on the future smile scenarios assumed possible for time T_1.

Today's ΔT-smile, which is observable in the market, will be denoted by $\hat{\sigma}_0(x)$. If we make the assumption that the future ΔT-smile, which prevails in the market at time T_1, will be identical to today's smile (that is the case, for example, in any pure Levy model), we obtain the present value of the contract as

$$
e^{-rT_1} \cdot V_{T_1}^H(\hat{\sigma}_0(x)) \quad (4)
$$

where we have used today's ΔT-smile $\hat{\sigma}_0(x)$ in formula (3).

Assume now that we recognize the uncertainty in the future smile and consider three possible scenarios: the smile moves up to $\hat{\sigma}_u(x)$, stays the same at $\hat{\sigma}_0(x)$ or moves down to $\hat{\sigma}_d(x)$ – with probabilities p_u, p_0 and p_d respectively. The value of the contract is now computed as

$$
e^{-rT_1} \cdot \left[p_u \cdot V_{T_1}^H(\hat{\sigma}_u(x)) + p_0 \cdot V_{T_1}^H(\hat{\sigma}_0(x)) + p_d \cdot V_{T_1}^H(\hat{\sigma}_d(x))\right]. \quad (5)
$$

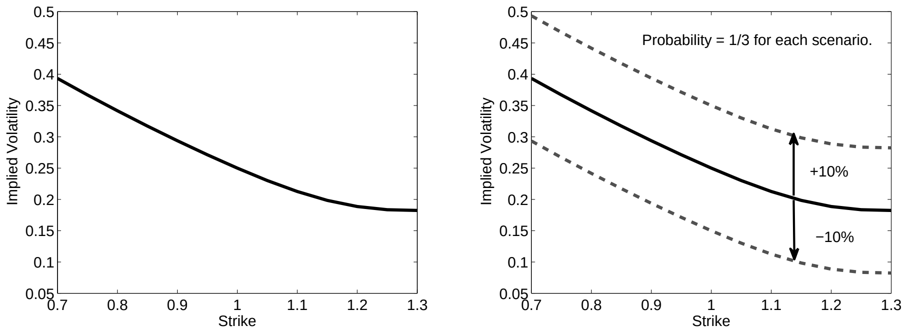

Figure 1: Comparison of two 3m-smile behaviors: (Left) the future 3m-smile assumed identical to today's 3m smile, (Right) the future 3m-smile assumed to take on 3 possible realizations (shifted up by 10 volatility points, remains the same and shifted down by 10 volatility points) with equal probabilities 1/3.

An interesting question is how the valuation without volatility-of-volatility in (4) compares to the valuation with volatility-of-volatility in (5).We next consider a simple numerical example. The left panel of Figure (1) shows the three-months, $\Delta T = 0.25$, S&P500 smile from July 31 2009; assume this is today's observed smile, denoted above by $\hat{\sigma}_0(x)$. With volatility-of-volatility, we assume three possible smile shifts: up by 10 volatility points ($\hat{\sigma}_u(x) = \hat{\sigma}_0(x) + 0.1$), constant and down 10 volatility points ($\hat{\sigma}_d(x) = \hat{\sigma}_0(x) - 0.1$) each with equal probability $\frac{1}{3}$. Remaining parameters are taken $T_1 = 0.25$, $T_2 = T_1 + \Delta T = 0.5$, interest rate $r = 0.4\%$ and dividend yield $\delta = 1.9\%$. We obtain the (undiscounted) contract value, without vol-of-vol, at 0.0863 and the value, with vol-of-vol, at $\frac{1}{3} \cdot (0.1727 + 0.0863 + 0.0313) = 0.0968$, for a relative difference of approximately 12.17%. We emphasize that, in both cases, the *expected smile is the same*; note that $\frac{1}{3} \cdot (\hat{\sigma}_u(x) + \hat{\sigma}_0(x) + \hat{\sigma}_d(x)) = \hat{\sigma}_0(x)$. Therefore, the significant valuation difference stems entirely from the volatility-of-volatility. We conclude that, a model which does not properly reflect the stochasticity of the future smile can severely misprice this product.

Let us now consider the valuation of a slightly more complicated contract, whose payoff at time T_2 is given by

$$\left(\frac{1}{T_2 - T_1} \cdot \log^2 \left(\frac{S_{T_2}}{S_{T_1}} \right) - \sigma_K^2 \right)_+$$

and which resembles (albeit remotely) an option on realized variance with volatility strike $\sigma_K > 0$. As before, we begin by determining the value of the contract at time T_1. This payoff can be decomposed as

$$\left(\frac{1}{T_2 - T_1} \cdot \log^2 \left(\frac{S_{T_2}}{S_{T_1}} \right) - \sigma_K^2 \right) \cdot \left(1_{S_{T_2} < S_{T_1} e^{-\sigma_K \sqrt{T_2 - T_1}}} + 1_{S_{T_2} > S_{T_1} e^{\sigma_K \sqrt{T_2 - T_1}}} \right)$$

and we let

$$
\begin{aligned}
H_L(x) &= \left(\frac{1}{T_2-T_1}\cdot\log^2\left(\frac{x}{S_{T_1}}\right)-\sigma_K^2\right)\cdot 1_{x<S_{T_1}e^{-\sigma_K\sqrt{T_2-T_1}}} \\
H_R(x) &= \left(\frac{1}{T_2-T_1}\cdot\log^2\left(\frac{x}{S_{T_1}}\right)-\sigma_K^2\right)\cdot 1_{x>S_{T_1}e^{\sigma_K\sqrt{T_2-T_1}}}.
\end{aligned}
$$

The function $H_L(x)$ is twice differentiable on $(0,\infty)\backslash\{S_{T_1}e^{-\sigma_K\sqrt{T_2-T_1}}\}$ with left and right derivatives at $S_{T_1}e^{-\sigma_K\sqrt{T_2-T_1}}$ given by

$$
\begin{aligned}
\frac{\partial H_L^-}{\partial x}\left(S_{T_1}e^{-\sigma_K\sqrt{T_2-T_1}}\right) &= \frac{-2\sigma_K}{S_{T_1}\sqrt{T_2-T_1}e^{-\sigma_K\sqrt{T_2-T_1}}} \\
\frac{\partial H_L^+}{\partial x}\left(S_{T_1}e^{-\sigma_K\sqrt{T_2-T_1}}\right) &= 0.
\end{aligned}
$$

Therefore, by applying to $H_L(x)$ the statement (2) of the spanning formula, we obtain

$$
\begin{aligned}
H_L(x) &= \frac{2\sigma_K}{S_{T_1}\sqrt{T_2-T_1}e^{-\sigma_K\sqrt{T_2-T_1}}}\cdot\left(S_{T_1}e^{-\sigma_K\sqrt{T_2-T_1}}-x\right)_+ \\
&\quad+\int_0^{S_{T_1}e^{-\sigma_K\sqrt{T_2-T_1}}}\frac{2}{K^2(T_2-T_1)}\left(1-\log\left(\frac{K}{S_{T_1}}\right)\right)\cdot(K-x)_+\,dK.
\end{aligned}
$$

After proceeding analogously with the function $H_R(x)$, we finally obtain that the value of the contract at the future time T_1 will be given by

$$
\begin{aligned}
V_{T_1}^H &= \frac{2\sigma_K}{S_{T_1}\sqrt{T_2-T_1}e^{-\sigma_K\sqrt{T_2-T_1}}}\cdot P(S_{T_1},S_{T_1}e^{-\sigma_K\sqrt{T_2-T_1}},T_2-T_1) \\
&\quad+\frac{2\sigma_K}{S_{T_1}\sqrt{T_2-T_1}e^{\sigma_K\sqrt{T_2-T_1}}}\cdot C(S_{T_1},S_{T_1}e^{\sigma_K\sqrt{T_2-T_1}},T_2-T_1) \\
&\quad+\int_0^{S_{T_1}e^{-\sigma_K\sqrt{T_2-T_1}}}\frac{2}{K^2(T_2-T_1)}\left(1-\log\left(\frac{K}{S_{T_1}}\right)\right)\cdot P(S_{T_1},K,T_2-T_1)dK \\
&\quad+\int_{S_{T_1}e^{\sigma_K\sqrt{T_2-T_1}}}^{\infty}\frac{2}{K^2(T_2-T_1)}\left(1-\log\left(\frac{K}{S_{T_1}}\right)\right)\cdot C(S_{T_1},K,T_2-T_1)dK.
\end{aligned}
$$

As before, making the change of variable $K=x\cdot S_{T_1}$ and using the Black-Scholes implied volatility-by-moneyness smile $\hat\sigma(x)$ prevailing in the market at time T_1, we obtain

$$
\begin{aligned}
V_{T_1}^H &= \frac{2\sigma_K}{\sqrt{T_2-T_1}e^{-\sigma_K\sqrt{T_2-T_1}}}\cdot P^{BS}\left(1,e^{-\sigma_K\sqrt{T_2-T_1}};\hat\sigma\left(e^{-\sigma_K\sqrt{T_2-T_1}}\right),T_2-T_1\right) \\
&\quad+\frac{2\sigma_K}{\sqrt{T_2-T_1}e^{\sigma_K\sqrt{T_2-T_1}}}\cdot C^{BS}\left(1,e^{\sigma_K\sqrt{T_2-T_1}};\hat\sigma\left(e^{\sigma_K\sqrt{T_2-T_1}}\right),T_2-T_1\right) \\
&\quad+\int_0^{e^{-\sigma_K\sqrt{T_2-T_1}}}\frac{2}{x^2(T_2-T_1)}(1-\log(x))\cdot P^{BS}(1,x;\hat\sigma(x),T_2-T_1)dx \\
&\quad+\int_{e^{\sigma_K\sqrt{T_2-T_1}}}^{\infty}\frac{2}{x^2(T_2-T_1)}(1-\log(x))\cdot C^{BS}(1,x;\hat\sigma(x),T_2-T_1)dx.
\end{aligned}
$$

Again, we notice that the value of the contract at time T_1 depends *only* on the ΔT-smile which will prevail in the market at time T_1; in particular, note that the value does not depend on the future stock price S_{T_1}. Similar to our earlier comparison, we consider the two smile behaviors depicted in Figure (1): (I) the ΔT-smile remains identical to today's smile and (II) the smile can shift up/down by 10 volatility points around today's smile. The two valuations are then given by formulas (4) and (5) with $V^H_{T_1}$ as above. Using $\sigma^2_K = 0.0968$ (the value of the previous contract), we obtain the (undiscounted) price, without vol-of-vol, at 0.044 and, with vol-of-vol, at $\frac{1}{3}(0.1161 + 0.044 + 0.0091) = 0.0564$ — for a relative difference of approximately 28.18%! As before, the expected smile is the same in both cases and, therefore, the pricing difference comes entirely from the volatility-of-volatility.

Both contracts considered so far had a substantially higher value *with* vol-of-vol than *without* vol-of-vol. This is explained by their positive convexity in volatility. Specifically, in our setting, the value $V^H_{T_1}(\hat{\sigma}(x))$ was convex in the level of the smile $\hat{\sigma}(x)$ and thus the average computed in equation (5) across the three possible smiles is larger than the value computed with the expected smile in equation (4). The importance of vol-of-vol is greater, the more volatility convexity a product has. In practice, this sensitivity is usually called Volga which, in turn, is just short-hand for Volatility Gamma.

As expected, different products can have vastly different Volgas. As another example, let us consider a contract whose payoff at time T_2 is

$$\left(\frac{S_{T_2}}{S_{T_1}} - 1\right)_+$$

i.e. a forward-started at-the-money call. It it straightforward to see that the value, at time T_1, of this contract is $C^{BS}(1, 1, \hat{\sigma}(1), T_2 - T_1)$, where $\hat{\sigma}(1)$ is the at-the-money implied Black-Scholes volatility of maturity ΔT prevailing in the market at time T_1. Proceeding as before, we compare the value without vol-of-vol $C^{BS}(1, 1, \hat{\sigma}_0(1), 0.25) = 4.782\%$ and the value with vol-of-vol

$$\begin{aligned} & \frac{1}{3} \cdot \left(C^{BS}(1, 1, \hat{\sigma}_u(1), 0.25) + C^{BS}(1, 1, \hat{\sigma}_0(1), 0.25) + C^{BS}(1, 1, \hat{\sigma}_d(1), 0.25)\right) \\ = \; & \frac{1}{3} \cdot (6.765\% + 4.782\% + 2.797\%) = 4.781\% \end{aligned}$$

and observe that the two valuations are essentially identical. This is explained by the fact that at-the-money options are almost *linear* in volatility i.e. have a Volga close to zero[1]. Figure (2) shows the Volga of European vanilla options across strikes. Indeed, we notice that ATM options have little

[1] We remark that it can, in fact, be slightly negative depending on the sign of $(r - \delta)^2 - \frac{\sigma^4}{4}$.

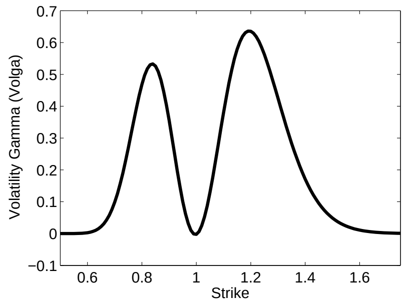

Figure 2: Volatility Gamma (Volga) of European vanilla options as a function of strike, for a Black-Scholes volatility of 25% and maturity 3-months.

Volga and that Volga peaks in a region OTM before dying off for far-OTM options. If we consider an OTM forward-started call with payoff

$$\left(\frac{S_{T_2}}{S_{T_1}} - 1.25\right)_+$$

by repeating the calculations above, we obtain a price without vol-of-vol of about 2.23 bps whereas the price with vol-of-vol is about 12.95 bps. Unlike the ATM case, vol-of-vol now has a substantial impact on valuation.

3 Conclusion

All the elementary payoffs that we have been considering in this short account appear, either explicitly or implicitly, in many types of exotic and structured products. Among these, we mention variance derivatives and the different variations of locally/globally, floored/capped, arithmetic/geometric cliquets. As noted in Eberlein, Madan (2009), the market for such products has been on an exponential growth trend. Therefore, for dealers pricing these products proper modeling of the volatility-of-volatility is of major importance. Bergomi (2005, 2008) proposes a forward-started modeling approach which allows direct control of the future smiles; a version which includes jumps is also given in Drimus (2010). In addition to pricing, the monitoring and risk-management of the Volatility Gamma (or Volga) becomes critical for an exotics book, as it drives the profit & loss of the daily rebalancing of the Vega. A further discussion of the Volga and Vanna[2], in a stochastic volatility model, can be found in Drimus (2011).

[2]The change in Delta w.r.t. a change in volatility $\frac{\partial \Delta}{\partial \sigma}$.

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